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Uniform Convergence of Convex Optimization Problems*

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INTRODUCTION

The study of convergence of convex sets and convex functions and related operators has received increasing attention since the mid-1960s. Such work has been motivated by efforts toward successive approximation schemes in a wide variety of areas, including statistics, variational inequalities, approximation theory, convex optimization, control theory, and mathematical programming.

Wisman [7, 8] was the first to introduce a new type of convergence of convex functions, viz., "epigraph convergence," in finite dimensions. Uberto Mosco [4] studied the convergence of convex functions using the notion of epigraph convergence in the context of variational problems and convex optimization. Recently, Bergstrom [2] has investigated further the notion of epigraph convergence and applied it to network optimization and convex programming.

In this paper, we deviate from the above notion of epigraph convergence and study the convergence of convex optimization problems under the familiar notion of uniform convergence. In mathematical programming, the interest in optimality conditions and other related questions is mainly focused on problems in finite dimensional spaces. The optimal control point of view has motivated many generalizations to infinite dimensional spaces. Hence, with the hope of future applications, we develop the theory in a more general setting.

The plan of the paper is as follows. In Section 1, we develop the necessary preliminaries that will be used in the paper, for the sake of completeness, and introduce two new definitions. In Section 2, under the familiar notion of uniform convergence, we prove that a sequence of convex optimization problems converge to a convex optimization problem. The connections of this convergence of optimization problems with two of the main operations of convex analysis: Fenchel transform and subdifferentiation are investigated in Sections 3 and 4.

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1. Preliminaries

Let X and Y be two locally convex spaces and Y also an ordered vector space with the positive cone K. That is, for $x, y \in Y$, $x \leqslant y$ iff $y - x \in K$. Then K defines an order relation "$\leqslant$" on Y with respect to which Y is an ordered vector space with positive cone K. If Y is R, the positive cone is $[0, \infty)$.

We need the following definitions which can be found in [5].

DEFINITIONS. Suppose that Y is an ordered topological vector space and that K is a positive cone in Y.

(i) A subset A of Y is said to be *full* if $a \leqslant c \leqslant b$, $a, b \in A$ implies $c \in A$.

(ii) The positive cone K is said to be *normal* for the topology in Y if there is a neighborhood basis of θ for the topology on Y consisting of full sets.

(iii) A map $f: X \to Y$ is said to be *convex* if

$$tf(x) + (1-t)f(y) - f(tx + (1-t)y) \in K$$

for all $x, y \in X$, $0 \leqslant t \leqslant 1$.

Let X^* be the topological dual space of X. That is, X^* is the set of all continuous linear functionals on X.

(iv) An element $x^* \in X^*$ is said to be a *subgradient* of a functional $f: X \to R$ at x_0 if

$$x^*(x - x_0) \leqslant f(x) - f(x_0)$$

for every $x \in X$. The set of all subgradients of f at x_0 is called the *subdifferential* of f at x_0 and is denoted by $\partial f(x_0)$. The functional f is said to be *subdifferentiable* at x_0 if $\partial f(x_0)$ is non-empty.

(v) Let $f: X \to R$ be any functional. Then the *Fenchel transform* of f is the functional $f^*: X^* \to R$ defined by

$$f^*(x^*) = \operatorname*{Sup}_{x \in X} \{ \langle x^*, x \rangle - f(x) \}.$$

f^* is also called the *conjugate* functional of f. It is easy to see that f^* is always convex and lower-semi-continuous. The reader is referred to Rockafellar [6] and Ekeland and Temam [3] for more details.

The following definitions can be seen in [4].

DEFINITIONS. Let $\{A_n\}$ be a sequence of subsets of a locally convex space X.

(i) *The strong limit infimum of* A_n, denotes by $s - \underline{\lim}\, A_n$, is defined by

$$s\text{-}\underline{\lim}\, A_n = \{v \in X\text{: there exists a sequence } v_n \text{ such that } v_n \in A_n \text{ and } v_n \to v\}.$$

(ii) *The weak limit infimum of* A_n, denoted by $w - \underline{\lim}\, A_n$, is defined by

$$w - \underline{\lim}\, A_n = \{v \in X\text{: there exists a sequence } v_n \text{ such that } v_n \in A_n \text{ and } v_n \to v \text{ weakly}\}.$$

2. CONVERGENCE THEOREMS

Let X and Y be locally convex spaces, and let Y be also an ordered vector space with a normal order cone. Let $f\colon X \to Y$ be a continuous, convex map. We consider the convex optimization problem

$$\alpha = \operatorname*{Inf}_{x \in X} f(x). \tag{P}$$

Let $S = \{x \in X : f(x) = \alpha\}$. It may not be easy to solve problem (P). We approximate f by a sequence $\{f_n\}$ of continuous, convex maps frm X to Y. Corresponding to each f_n, we formulate the convex optimization problem.

$$\alpha_n = \operatorname*{Inf}_{x \in X} f_n(x). \tag{P$_n$}$$

Let $S_n = \{x \in X : f_n(x) = \alpha_n\}$. We shall assume that S, S_n, $n = 1, 2, \ldots$, are nonempty subsets of X.

By a proper choice of the f_n's, it may be easy to solve the problems (P_n). The following questions arise naturally. (i) Does the sequence $\{\alpha_n\}$ converge to α? (ii) Does the sequence $\{S_n\}$ of optimality sets (sets of solutions) of (P_n) converge (in some sense) to the optimality set S of problem (P)? These, and other related questions, will be the subject of discussion of this paper.

We introduce the following definitions.

DEFINITIONS. (1) Any sequence $\{x_n\}$, such that $x_n \in S_n$ for $n = 1, 2, \ldots$, will be called a *sequence of solutions* of the family (P_n).

(2) Th sequence of problems (P_n) will be said to *converge* to (P) if

(i) $\alpha_n \to \alpha$ and

(ii) if $\{x_n\}$ is a sequence of solutions of (P_n) converging to $x_0 \in X$, then $x_0 \in S$; that is, $s-\underline{\lim}\, S_n \subset S$.

The main objective of this paper is to obtain conditions under which the sequence of problems (P_n) converges to problem (P), and the establish their connections under two of the main operations of convex analysis: the Fenchel transform and subdifferentiation.

THEOREM 2.1. *Let $f_n \to f$ uniformly. Then the sequence of problems* (P_n) *converges to problem* (P).

Proof. (i) Since $f_n \to f$ uniformly, given any full neighborhood V of the origin in Y, there is an $n_0 \in N$ such that

$$f_n(x) - f(x) \in V$$

for all $n \geqslant n_0$, and for all $x \in X$. Then, we have

$$f_n(x_0) - f(x_0) \in V$$

for all $n \geqslant n_0$ and

$$f_n(x_n) - f(x_n) \in V$$

for all $n \geqslant n_0$. That is,

$$f_n(x_0) - \alpha \in V,$$

for all $n \geqslant n_0$, and $\alpha_n - f(x_n) \in V$, for all $n \geqslant n_0$. Also $\alpha_n - f(x_n) \leqslant \alpha_n - \alpha \leqslant f_n(x_0) - \alpha$, since $\alpha_n \leqslant f_n(x)$ for all $x \in X$ and $\alpha \leqslant f(x)$ for all $x \in X$. Since Y has a normal order cone, it follows that $\alpha_n - \alpha \in V$ for all $n \geqslant n_0$. Hence $\alpha_n \to \alpha$.

(ii) Let $\{x_n\}$ be a sequence of solutions of the problems (P_n) such that $x_n \to x_0$. Let V be a full, symmetric neighborhood of the origin in Y. Since f is continuous, we have $f(x_n) \to f(x_0)$. Hence there is an $n_1 \in N$ such that

$$f(x_n) - f(x_0) \in \tfrac{1}{3} V \tag{1}$$

for all $n \geqslant n_1$. By (i), $\alpha_n \to \alpha$. Hence there is an $n_2 \in N$ such that

$$\alpha_n - \alpha \varepsilon \tfrac{1}{3} V \tag{2}$$

for all $n \geqslant n_2$. Since $f_n \to f$ uniformly, there is an $n_3 \in N$ such that

$$f_n(x) - f(x) \in \tfrac{1}{3} V \tag{3}$$

for all $n \geqslant n_3$ and for all $x \in X$. Let $n_0 = \max\{n_1, n_2, n_3\}$. From (3), $f_{n_0}(x_{n_0}) - f(x_{n_0}) \in \frac{1}{3}V$. That is,

$$\alpha_{n_0} - f(x_{n_0}) \in \tfrac{1}{3}V. \tag{4}$$

From (2), $\alpha_{n_0} - \alpha \in \frac{1}{3}V$ or

$$\alpha - \alpha_{n_0} \in \tfrac{1}{3}V. \tag{5}$$

From (1), we have

$$f(x_{n_0}) - f(x_0) \in \tfrac{1}{3}V. \tag{6}$$

From (4), (5) and (6), we have $\alpha - f(x_0) \in V$. Hence $x_0 \in S$. That is, $s - \underline{\lim}\, S_n \subset S$. Hence the theorem.

COROLLARY. *If $\bigcup_{n=1}^{\infty} S_n$ is compact, then every sequence of solutions of* (P_n) *has a subsequence which converges to a solution of* (P).

Note. The question that naturally arises is whether $s - \underline{\lim}\, S_n$ is, in fact, equal to S. In other words, if x_0 is a solution of (P), is it necessarily the limit of a sequence of solutions of (P_n)? That such is not generally the case can be seen by the following example.

EXAMPLE. Consider the following real-valued functions defined on the real line.

$$
\begin{aligned}
f(x) &= x^2 && \text{for} \quad |x| > 1,\\
&= 1 && \text{for} \quad |x| \leqslant 1.\\
f_n(x) &= x^2 && \text{for} \quad |x| > 1,\\
&= 1 - (1/n)(x+1) && \text{for} \quad -1 \leqslant x \leqslant 0,\\
&= 1 + (1/n)(x-1) && \text{for} \quad 0 \leqslant x \leqslant 1.
\end{aligned}
$$

It is easy to see that $\{f_n\}$ is a sequence of continuous convex functions, converging uniformly to the continuous convex function f. It is also clear that $S_n = \{0\}$ for $n = 1, 2, \ldots$, while $S = [-1, 1]$. Hence $s - \underline{\lim}\, S_n = \{0\}$ which is a proper subset of S.

The following theorem gives sufficient conditions for $s - \underline{\lim}\, S_n$ to be equal to S.

THEOREM 2.2. *Suppose X is a finite dimensional space, and Y is R, and suppose f and f_n, $n = 1, 2, \ldots$, are also strictly convex. Let $f_n \to f$ uniformly. Then $s - \underline{\lim}\, S_n = S$.*

To prove the theorem, we need the following lemma.

LEMMA. *Let X be a finite dimensional space, and let $f: X \to R$ be a strictly convex function attaining its minimum at x_0. Then, if V is a neighborhood of x_0, there is an $\varepsilon > 0$ such that*

$$V \supset f^{-1}[f(x_0), f(x_0) + \varepsilon).$$

Proof of the lemma. Suppose, on the contrary, for each $\varepsilon > 0$, there is $x_\varepsilon \in f^{-1}[f(x_0), f(x_0) + \varepsilon) \setminus \bar{V}$, where $\bar{V}$ denotes the closure of V. Let the line joining x_ε to x_0 meet the boundary of V at y_ε and z_ε. Then, one of y_ε and z_ε is a convex linear combination of x_0 and x_ε. Let it be y_ε (say). Then $y_\varepsilon = tx_0 + (1-t)x_\varepsilon$, where $0 < t < 1$. By strict convexity of f,

$$f(y_\varepsilon) < tf(x_0) + (1-t)f(x_\varepsilon).$$

Therefore

$$\alpha \leqslant f(y_\varepsilon) < t\alpha + (1-t)(\alpha + \varepsilon) = \alpha + (1-t)\varepsilon < \alpha + \varepsilon.$$

Thus, for each $n = 1, 2, \ldots$, we can find $y_n \in$ boundary of V such that

$$\alpha \leqslant f(y_n) < \alpha + \frac{1}{n}. \tag{7}$$

Since X is finite dimensional, the boundary of V is compact, and so $\{y_n\}$ has a convergent subsequence $\{y_{n_k}\}_{k=1}^{\infty}$ converging to $y_0 \in$ boundary of V. Hence $y \neq x_0$. From (7), it is clear that

$$f(y_{n_k}) \to \alpha. \tag{8}$$

Since $y_{n_k} \to y_0$,

$$f(y_{n_k}) \to f(y_0). \tag{9}$$

From (8) and (9), $f(y_0) = \alpha$. But $f(x_0) = \alpha$. This contradicts the fact that a strictly convex function attains its infimum at a unique point. Hence there is an $\varepsilon > 0$ such that

$$V \supset f^{-1}[f(x_0), f(x_0) + \varepsilon).$$

Proof of Theorem. Let $x \in S$. Since $f_n \to f$ uniformly, we have, given $\varepsilon > 0$, that there is an $n_0 \in N$ such that $|f_n(x) - f(x)| < \varepsilon$, for all $n \geqslant n_0$ and for all $x \in X$. Hence $f(x_n) < f_n(x_n) + \varepsilon$, for all $n \geqslant n_0$. That is, $f(x_n) < \alpha_n + \varepsilon/2$, for all $n \geqslant n_0$. Thus

$$\alpha = f(x_0) \leqslant f(x_n) < \alpha_n + \varepsilon, \tag{10}$$

for all $n \geqslant n_0$. Since $\alpha_n \to \alpha$, there is an $n_1 \in N$ such that $\alpha_n < \alpha + \varepsilon/2$, for all $n \geqslant n_1$. Hence, by (10),

$$f(x_0) \leqslant f(x_n) < \alpha + \varepsilon,$$

for all $n \geqslant n_2 = \max\{n_0, n_1\}$. That is,

$$f(x_0) \leqslant f(x_n) < f(x_0) + \varepsilon,$$

for all $n \geqslant n_2$. Therefore,

$$x_n \in f^{-1}[f(x_0), f(x_0) + \varepsilon)$$

for all $n \geqslant n_2$.

Let V be a neighborhood of x_0 in X. By Lemma, there exists an $\varepsilon > 0$ such that

$$f^{-1}[f(x_0), f(x_0) + \varepsilon) \subset V.$$

Hence $x_n \in V$, for all $n \geqslant n_2$. Thus, $x_n \to x_0$ and $x_0 \in s - \underline{\lim}\, S_n$.

Remark. The lemma, used in the proof of Theorem 2.2, is not true if X is infinite dimensional. For instance, consider the following example.

EXAMPLE. Let $f\colon l_2 \to R$ be defined by

$$f(x) = \sum_1^{\infty} \frac{x_i^2}{i},$$

where $x \equiv (x_n) \in l_2$. It is clear that f is strictly convex and attains its minimum at 0. Consider the neighborhood $B(0, 1) = \{y \in l_2 : \|y\| < 1\}$ of 0. Given any $\varepsilon > 0$, let n be a positive integer so that $1 < n \cdot \varepsilon$. Let x_n be a real number satisfying $1 < x_n^2 < n \cdot \varepsilon$. Then the element $x \equiv (x_n)$, where $x_m = 0$ for $m \neq n$, belongs to $f^{-1}[f(0), f(0) + \varepsilon)$ but $x \notin B(0, 1)$.

3. Connections with Fenchel Transform

THEOREM 3.1. *Let $\{f_n\}$ be a sequence of continuous convex functionals defined on a locally convex space X, converging uniformly to a continuous convex functional f. Then $f_n^* \to f^*$ uniformly on the dual space X^*.*

Proof. By the definition of Fenchel transform, we have

$$f_n^*(x^*) = \sup_{x \in X} \{\langle x, x^* \rangle - f_n(x)\} \tag{11}$$

and

$$f^*(x^*) = \sup_{x \in X} \{\langle x, x^* \rangle - f(x)\}. \tag{12}$$

Let $\varepsilon > 0$ be given. Let x^* be an arbitrary element of X^*, and let n be any positive integer. From (11), there exists $x_1 \in X$ such that

$$f_n^*(x^*) - \varepsilon/2 < \langle x_1, x^* \rangle - f_n(x_1). \tag{13}$$

From (12), there exists $x_2 \in X$ such that

$$f^*(x^*) - \varepsilon/2 < \langle x_2, x^* \rangle - f(x_2). \tag{14}$$

From (13) and (14), and from the definition of Fenchel transform, we have

$$\langle x_2, x^* \rangle - f_n(x_2) - \varepsilon/2 \leqslant f_n^*(x^*) - \varepsilon/2 < \langle x_1, x^* \rangle - f_n(x_1)$$

and

$$\langle x_1, x^* \rangle - f(x_1) - \varepsilon/2 < f_n^*(x^*) - \varepsilon/2 < \langle x_2, x^* \rangle - f(x_2).$$

Hence

$$f(x_2) - f_n(x_2) - \varepsilon/2 < f_n^*(x^*) - f^*(x^*) < f(x_1) - f_n(x_1) + \varepsilon/2. \tag{15}$$

Since $f_n \to f$ uniformly, there is $n_0 \in N$ such that $|f(x) - f_n(x)| < \varepsilon/2$ for all $n \geqslant n_0$, and for all $x \in X$. That is,

$$-\varepsilon/2 < f_n(x) - f(x) < \varepsilon/2 \tag{16}$$

for all $n \geqslant n_0$, and for all $x \in X$. From (15) and (16), we see that

$$-\varepsilon < f_n^*(x^*) - f^*(x^*) < \varepsilon,$$

for all $n \geqslant n_0$ and for all $x^* \in X^*$. Hence $f_n^* \to f^*$ uniformly on X^*.

Remark. It is interesting to note that the first part of Theorem 2.1 may be obtained as a corollary to the above theorem.

Alternate proof of Theorem 2.1(i).

$$\begin{aligned} \alpha_n &= \operatorname*{Inf}_{x \in X} f_n(x) = -\operatorname*{Sup}_{x \in X} \{\langle x, 0 \rangle - f_n(x)\} \\ &= -f_n^*(0), \end{aligned}$$

and

$$\alpha = \operatorname*{Inf}_{x \in X} f(x) = -f^*(0).$$

Since $f_n^* \to f^*$ uniformly on X^*, $f_n^* \to f$ pointwise, and $f_n^*(0) \to f^*(0)$. Thus $\alpha_n \to \alpha$.

4. Connections with Subdifferentiation

Suppose that each f_n and f is a strictly convex and continuous functional on a locally convex space X and that $f_n \to f$ uniformly. Let $S_n = \{x_n\}$, $n = 1, 2, \ldots$, and $S = \{x_0\}$.

Since x_n is a solution (P_n) and x_0 is a solution of problem (P), we have, as a consequence of the definition of subdifferential, $0 \in \partial f_n(x_n)$, $n = 1, 2, \ldots$, and $0 \in \partial f(x_0)$.

Suppose $\{x_n^*\}$ is a sequence in X^* such that $x_n^* \in \partial f_n(x_n)$ for each n, and suppose $x_n^* \to x_0^*$ in some sense. It is natural to ask whether x_0^* is in $\partial f(x_0)$. This is so, in the following situation.

Theorem 4.1. *Let X be a Banach space. Suppose that $\bigcup_{n=1}^{\infty} \{x_n\}$ is compact. Let $x_n^* \in \partial f_n(x_n)$, $n = 1, 2, \ldots$, and let $x_n^* \to x_0^*$ weakly. Then $x_0^* \in \partial f(x_0)$. In other words, $w - \underline{\lim}\, \partial f_n(x_n) \subset \partial f(x_0)$.*

Proof. Since $\bigcup_{n=1}^{\infty} \{x_n\}$ is compact, the sequence $\{x_n\}$ of solutions of the family (P_n) has a convergent subsequence $\{x_{n_k}\}$. Also, if $x_{n_k} \to y$, then $y \in S$, by Theorem 2.1(ii). But S is the singleton $\{x_0\}$. Hence, $y = x_0$ and $x_{n_k} \to x_0$. Thus every convergent subsequence of $\{x_n\}$ converges to x_0. This shows that $x_n \to x_0$.

Since $x_n^* \to x_0^*$ weakly, we have

$$\langle x_n^*, x\rangle \to \langle x_0^*, x\rangle, \tag{17}$$

for all $x \in X$. Hence the sequence $\{|\langle x_n^*, x\rangle|\}$ is bounded, for each $x \in X$. By the unifirm boundedness principle [1], the sequence $\{\|x_n^*\|\}$ is bounded. Let $\|x_n^*\| \leqslant M$ for $n = 1, 2, \ldots,$

Now, consider

$$\begin{aligned}
&|\langle x_n^*, x_n\rangle - \langle x_0^*, x_0\rangle| \\
&\quad \leqslant |\langle x_n^*, x_n - x_0 > + \langle x_n^* - x_0^*, x_0 > | \\
&\quad \leqslant \|x_n^*\|\, \|x_n - x_0\| + |\langle x_n^*, x_0\rangle - \langle x_0^*, x_0\rangle| \\
&\quad \leqslant M\, \|x_n - x_0\| + |\langle x_n^*, x_0\rangle - \langle x_0^*, x_0\rangle|.
\end{aligned}$$

Since $\|x_n - x_0\| \to 0$ as $n \to \infty$, we have, by (17),

$$\langle x_n^* - x_n\rangle \to \langle x_0^*, x_0\rangle \tag{18}$$

as $n \to \infty$.

Since $x_n^* \in \partial f_n(x_n)$, $\langle x_n^*, x - x_n \rangle \leqslant f_n(x) - f_n(x_n)$ for each $x \in X$ and for every $n \in N$. That is,

$$\langle x_n^*, x \rangle - \langle x_n^*, x_n \rangle \leqslant f_n(x) - f_n(x_n) \tag{19}$$

for each $x \in X$ and for every $n \in N$.

Since $f_n \to f$ uniformly, we have, by Theorem 2.1(i),

$$f_n(x_n) = \alpha_n \to \alpha = f(x_0). \tag{20}$$

From (18) and (20), we obtain, by taking limit as $n \to \infty$ in (19),

$$\langle x_0^*, x \rangle - \langle x_0^*, x_0 \rangle \leqslant f(x) - f(x_0)$$

for each $x \in X$. That is, $x_0^* \in \partial f(x_0)$. Hence the theorem.

Remark. Under the conditions of the above theorem, we see that

$$s - \underline{\lim}\, \partial f_n(x_n) \subset w - \underline{\lim}\, \partial f_n(x_n) \subset \partial f(x_0).$$

However, $\partial f(x_0)$ need not be equal to $s - \underline{\lim}\, \partial f_n(x_n)$, as can be seen from the following example.

Example. We consider the following real-valued functions defined on the real line. Let

$$\begin{aligned} f(x) &= x^2 - 2x \qquad \text{for} \quad x \leqslant 0, \\ &= x^2 + 2x \qquad \text{for} \quad x \geqslant 0. \end{aligned}$$

Let f_n be the function whose graph is

$(x, x^2 - 2x)$, for $x \leqslant 1 - \sqrt{1 + 1/n}$,

a circular arc joining

$(1 - \sqrt{1 + 1/n}, 1/n)$ to $(-1 + 1/n, 1/n)$,

for $1 - \sqrt{1 + 1/n} \leqslant x \leqslant -1 + \sqrt{1 + 1/n}$,

$(x, x^2 + 2x)$, for $x \geqslant -1 + \sqrt{1 + 1/n}$,

where the circular arc is so chosen that $f_n(x) \geqslant 0$ for all x and so that f_n is strictly convex: Then, it is straightforward to see that f, f_n are continuous, strictly convex and that $f_n \to f$ uniformly. Also $x_n = 0$ for $n = 1, 2, \ldots$, and $x = 0$, so that $\bigcup_{n=1}^{\infty} x_n = \{0\}$ is compact.

Now, $\partial f_n(x_n) = \partial f_n(0) = \{0\}$ for $n = 1, 2, \ldots$, while $\partial f(x_0) = \partial f(0) = [-2, 2]$. Thus

$$\partial f(0) \not\subset s - \underline{\lim}\, \partial f_n(0).$$

THEOREM 4.2. *Let X be a locally convex space. Suppose that $\bigcup_{n=1}^{\infty} \{x_n\}$ is compact, $x_n^* \in \partial f_n(x_n)$, for $n = 1, 2, \ldots$, the family $\{x_n^*\}$ is equicontinuous at x_0, and $x_n^* \to x_0^*$ weakly. Then $x_0^* \in \partial f(x_0)$.*

Proof. Let $\varepsilon > 0$ be given. Since the family $\{x_n^*\}$ is equicontinuous at x_0, there is a neighborhood V of x_0 in X such that

$$|\langle x_n^*, y\rangle - \langle x_n^*, x_0\rangle| < \varepsilon/2 \tag{21}$$

whenever $y \in V$ and for $n = 1, 2, \ldots$.

Since $x_n \to x_0$ (as has been remarked in the proof of Theorem 4.1), there is an $n_1 \in N$ such that

$$x_n \in V \tag{22}$$

for all $n \geqslant n_1$. Hence from (21) and (22),

$$|\langle x_n^*, x_n\rangle - \langle x_n^*, x_0\rangle| < \varepsilon/2 \tag{23}$$

for all $n \geqslant n_1$.

Now,

$$\begin{aligned} |\langle x_n^*, x_n\rangle - \langle x_0^*, x_0\rangle| &\leqslant |\langle x_n^*, x_n\rangle - \langle x_n^*, x_0\rangle| \\ &\quad + |\langle x_n^*, x_0\rangle - \langle x_0^*, x_0\rangle|. \end{aligned}$$

The first term on the right-hand side of the inequality is less than $\varepsilon/2$ whenever $n \geqslant n_1$ by (23). The second term can also be made less than $\varepsilon/2$ for all large values of n, since $x_n^* \to x_0^*$ weakly. Thus

$$\langle x_n^*, x_n\rangle \to \langle x_0^*, x_0\rangle.$$

The rest of the proof is exactly on the same lines as in the proof of Theorem 4.1.

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